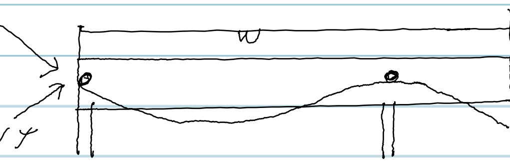
CH. 12.1 – 12.2 SLOPE + DISPLACEMENT BY INTEGRATION

SECTION 12.1 THE ELASTIC CURVE



$v(x)$ VERTICAL DISPLACEMENT

SMALL "v"

LOADS + MOMENTS CAUSE BENDING

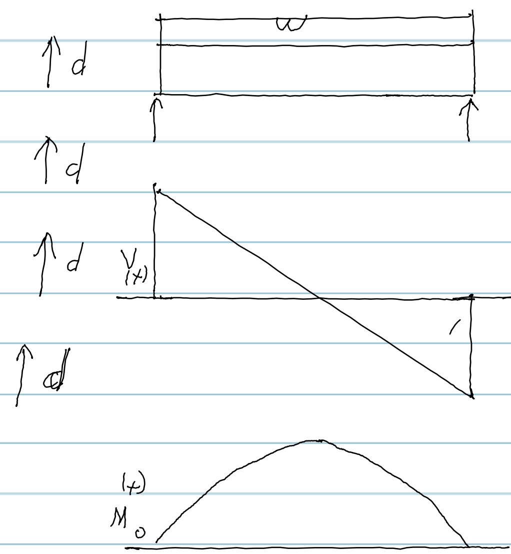
LOADS $\int\downarrow$

SHEAR $\int\downarrow$

MOMENT $\int\downarrow$

ANGLE OF BEAM $\int\downarrow$

VERTICAL POSITION OF BEAM



$$EI\frac{d^4v}{dx^4} = w(x)$$

$$EI\frac{d^3v}{dx^3} = V_s(x)$$

$$EI\frac{d^2v}{dx^2} = M(x)$$

$$EI\frac{dv}{dx} = \theta(x)$$

SOLVE FOR $v(x)$ USING $M(x)$

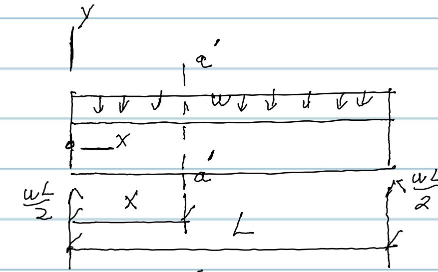
$$EI\frac{d^2v}{dx^2} = M(x)$$

$$EI\frac{dv}{dx} = \int M(x)$$

$$EI\,v(x) = \int\left[\int M(x)\right]$$

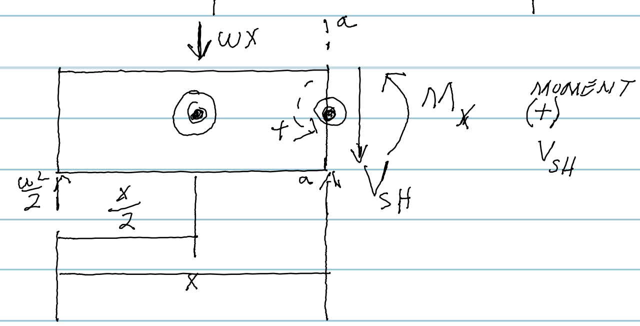
CH 12.1 – 12.2 ELASTIC CURVE (CONT.)

□ EXAMPLE SIMPLE BEAM W/ UNIFORM LOAD



□ FIND EXTERNAL REACTIONS

□ DRAW FBD @ C.S. "a-a"



□ $\sum M_{CENTROID} = 0$

$$-\left(\frac{wL}{2}\cdot x\right) + \left(wx\cdot\frac{x}{2}\right) + M_x = 0$$

□ $M_x = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$

USE ELASTIC CURVE EQUATION

$$EI\frac{d^2v}{dx^2} = M(x) \qquad \int EI\frac{d^2v}{dx^2} = \int M(x) = \int_0^x -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

□ * $EI\frac{dv}{dx} = -\frac{wx^3}{2\cdot 3} + \frac{wLx^2}{2\cdot 2} + C_1$

□ ** $EI\,v(x) = -\frac{wx^4}{2\cdot 3\cdot 4} + \frac{wLx^3}{2\cdot 2\cdot 3} + C_1x + C_2$

□ USING BOUNDARY CONDITIONS TO FIND CONSTANTS OF INTEGRATION

1) $x=0 \quad v(0)=0$ 2) $x=L \quad v(L)=0$

□ 1) $EI(v(0)) = -\frac{w\cdot 0^4}{2\cdot 3\cdot 4} + \frac{wL\,0^3}{2\cdot 2\cdot 3} + C_1\cdot 0 + C_2$

$EI\cdot 0 = C_2$ $\qquad C_2 = 0$

□ 2) $x=L \quad v(L)=0$

$$EI(v(L)) = -\frac{wL^4}{2\cdot 3\cdot 4} + \frac{wL\,L^3}{2\cdot 2\cdot 3} + C_1L$$

$0 =$

$$C_1 = \frac{wL^3}{2\cdot 3\cdot 4} - \frac{2wL^3}{2\cdot 2\cdot 3\cdot 2} = -\frac{wL^3}{24}$$

ANS: $v(x) = \frac{1}{EI}\cdot\left(-\frac{wx^4}{24} + \frac{wLx^3}{12} - \frac{wL^3x}{24}\right)$ GRAPH (CONT.)

CH. 12.1 – 12.2 ELASTIC CURVE (CONT.)

FIND MAXIMUM DEFLECTION – SIMPLE BEAM w/ UNIFORM LOAD

ELASTIC CURVE: $v(x) = \frac{1}{EI} \cdot \left(\frac{-wx^4}{24} + \frac{wLx^3}{12} - \frac{wL^3x}{24} \right)$

SET $\frac{dv}{dx} = 0$ SOLVE FOR X

$$\frac{dv}{dx} = \frac{1}{EI} \cdot \left(-\frac{wx^3}{6} + \frac{wLx^2}{4} - \frac{wL^3}{24} \right) = 0$$

$$-\frac{x^3}{6} + \frac{Lx^2}{4} - \frac{L^3}{24} = 0$$

MATHCAD SOLUTION

1) USE SPECIAL "=" CTRL + =
2) ENTER EQUATION w/ "="
✓ 3) HIGHLIGHT "X"
✓ 4) CLICK ON SYMBOLICS, THEN VARIABLE, THEN SOLVE
✓ 5) GET 3 SOLUTIONS $x = \frac{L}{2}$ $0 < x < L$

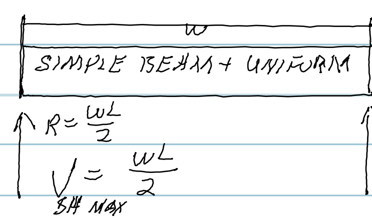
COULD USE SYMMETRY

SUBSTITUTE $x = \frac{L}{2}$ INTO ELASTIC CURVE $v(x)$

MAXIMUM DEFLECTION

$$v\left(\frac{L}{2}\right) = \frac{1}{EI} \cdot \left(-\frac{w\left(\frac{L}{2}\right)^4}{24} + \frac{wL\left(\frac{L}{2}\right)^3}{12} - \frac{wL^3\left(\frac{L}{2}\right)}{24} \right) = \frac{-5wL^4}{384EI}$$

LINEAR IN "w"



$M_b = \frac{wL^2}{8}$

$\Delta = \frac{-5wL^4}{384EI}$

CH. 12.3 DISCONTINUITY FUNCTIONS

ALTERNATE METHOD – ELASTIC CURVE

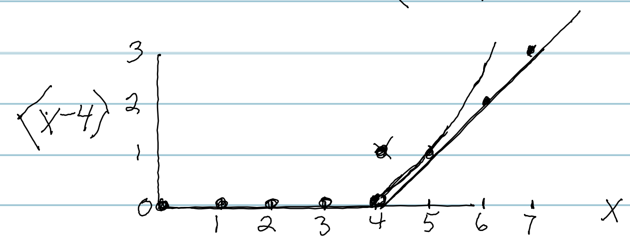
MACAULAY FUNCTION

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x<a \\ (x-a)^n & \text{for } x\geq a \end{cases}$$

WHEN	FUNCTION
$x-a<0$	0
$x-a\geq 0$	$(x-a)^n$

GRAPH $\langle x-4\rangle^1$

VIP $\langle -VAL\rangle = 0$



WRITE LOAD, SHEAR, OR MOMENT EQUATION DIRECTLY TABLE (12-2)

MOMENT TRANSFORMS DIRECTLY

FOR EACH POINT LOAD "P" PUT $P\langle x-a\rangle^1$ INTO MOMENT EQ

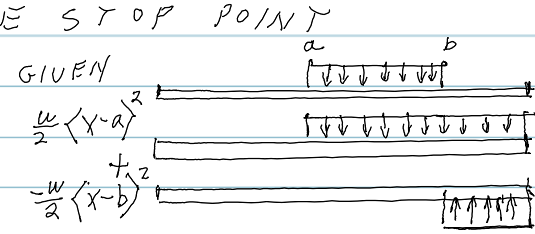
FOR EACH UNIFORM LOAD "w" PUT $\frac{w}{2}\langle x-a\rangle^2$ INTO MOMENT EQ.

FOR EACH MOMENT LOAD "M_0" PUT $M_0\langle x-a\rangle^0$ INTO MOMENT EQ.

FOR EACH ~~m~~ ~~$w = m(x-a)$~~ PUT $\frac{m}{6}\langle x-a\rangle^3$ " " "

NOTE: 1) "a" DISTANCE FROM LEFT END OF BEAM TO POINT WHERE LOAD, MOMENT IS APPLIED OF DISTRIBUTED FORCE STARTS.

2) TO STOP A DISTRIBUTED FORCE, A NEGATIVE "w" MUST BE APPLIED AT THE STOP POINT

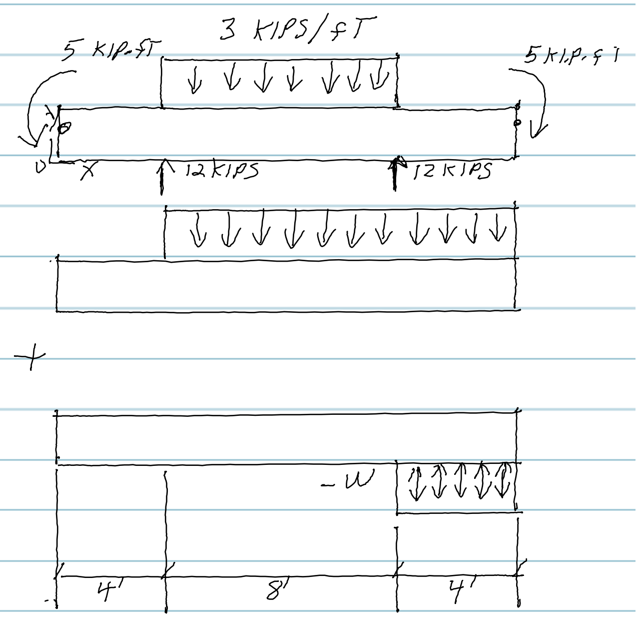


VIP 3) APPLIED MOMENTS ARE + IN CW DIRECTION

CH. 12.3 MACAULAY FUNCTIONS PROB. 12-35

GIVEN BEAM LOADED AS SHOWN

FIND ELASTIC CURVE



SOLUTION:

$$M(x) = -5\langle x-0\rangle^0 + 12\langle x-4\rangle^1 - \frac{3}{2}\langle x-4\rangle^2 + 12\langle x-12\rangle^1 + \frac{3}{2}\langle x-12\rangle^2$$

$$EI\frac{d^2v}{dx^2} = M(x) = -5 + 12\langle x-4\rangle^1 - \frac{3}{2}\langle x-4\rangle^2 + 12\langle x-12\rangle^1 + \frac{3}{2}\langle x-12\rangle^2$$

$$EI\frac{dv}{dy} = -5x + 6\langle x-4\rangle^2 - \frac{1}{2}\langle x-4\rangle^3 + 6\langle x-12\rangle^2 + \frac{1}{2}\langle x-12\rangle^3 + C_1$$

$$EI\,V(x) = -2.5x^2 + 2\langle x-4\rangle^3 - \frac{1}{8}\langle x-4\rangle^4 + 2\langle x-12\rangle^3 + \frac{1}{8}\langle x-12\rangle^4 + C_1 x + C_2$$

B.C

$V(4)=0$

$$0 = -2.5(4)^2 + 2\langle 4-4\rangle^3 - \frac{1}{8}\langle 4-4\rangle^4 + 2\langle 4-12\rangle^3 + \frac{1}{8}\langle 4-12\rangle^4 + C_1 \cdot 4 + C_2$$

$$① \quad 0 = -2.5\cdot 4^2 + 0 + 0 + 2\cancel{\langle -8\rangle^3}_{0} + \frac{1}{8}\cancel{\langle -8\rangle^4}_{0} + 4C_1 + C_2$$

$V(12)=0$

$$0 = -2.5(12)^2 + 2\langle 12-4\rangle^3 - \frac{1}{8}\langle 12-4\rangle^4 + 2\langle 8-12\rangle^3 + \frac{1}{8}\langle 8-12\rangle^4 + C_1 \cdot 12 + C_2$$

$$0 = -2.5(12)^2 + 2\langle +8\rangle^3 - \frac{1}{8}\langle +8\rangle^4 + 2\cancel{\langle -4\rangle^3}_{0} + \frac{1}{8}\cancel{\langle -4\rangle^4}_{0} + 12C_1 + C_2$$

$$② \quad 0 = -2.5\cdot 12^2 + 2\cdot 8^3 + 12C_1 + C_2$$

$C_1 = -24$ + $C_2 = 136$

$$EI\,V(x) = -2.5x^2 + 2\langle x-4\rangle^3 - \frac{1}{8}\langle x-4\rangle^4 + 2\langle x-12\rangle^3 + \frac{1}{8}\langle x-12\rangle^4 + (-24)x + 136$$

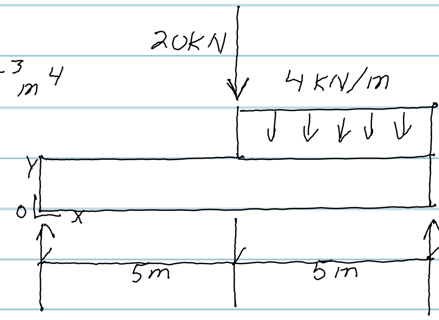
CH. 12-5 SUPERPOSITIONING

LOAD VS DEFLECTION - LINEAR

CALCULATE DEFLECTION FOR EACH LOAD USE APPENDIX "C" (813)

EXAMPLE - PROB 12-91

GIVEN: BEAM A36 STEEL, $I = 0.1457 \cdot 10^{-3} m^4$
LOADS + SUPPORTS AS SHOWN



FIND: DEFLECTION AT CTR. SPAN

SOLUTION: $E = 200\ GPa = 2 \cdot 10^{11} Pa$, $x_c = 5m$, $w = 4000 \frac{N}{m}$
$P = 20kN = 20{,}000\ N$, $L = 10m$

DEFLECTION FOR POINT LOAD @ CTR

$$V_{maxPL} = \frac{-PL^3}{48EI} = \frac{-2 \cdot 10^4 N \cdot (10m)^3}{48(2 \cdot 10^{11} Pa)(0.1457 \cdot 10^{-3} m^4)}$$

$$V_{maxPL} = -.0143\ m = \underline{\underline{-14.3\ mm.}}$$

DEFLECTION FOR UNIFORM LOAD ½ SPAN (END TO CTR.)

$$\underline{\underline{V_{maxu} = \frac{-wx}{384EI}}}\left(16x^3 - 24Lx^2 + 9L^3\right) \quad 0 \le x \le \frac{L}{2}$$

$$V_{mu} = \frac{(-4000\frac{N}{m})(5m)}{384(2 \cdot 10^{11} Pa)(.1457 \cdot 10^{-3} m^4)}\left(16 \cdot (5m)^3 - 24 \cdot 10m \cdot (5m)^2 + 9(10m)^3\right)$$

$$V_u = \underline{\underline{-8.939\ mm. = -0.008939\ m}}$$

DEFLECTION TOTAL @ CTR SPAN

$$V_{max} = 8.939mm + 14.3mm = \underline{\underline{23.2\ mm}}$$

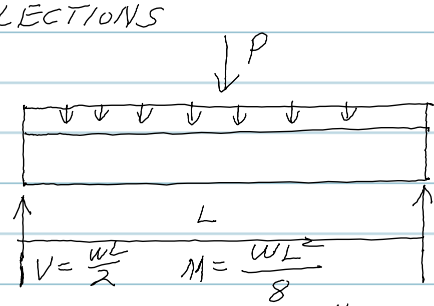
CH. 12.65 SUPERPOSITION (CONT.)

LINEAR RELATIONSHIP LOADS + DEFLECTIONS

SEPERATE LOADS FIND EACH DEFLECTION

ADD DEFLECTIONS @ POINT OF INTEREST

USE TABLE IN APPENDIX "C" (Pg. 813)



$$\Delta_u = V_{max} = \frac{-5wL^4}{384EI} \quad (Pg\ 89)$$

ADD POINT LOAD TO CTR OF BEAM

TOTAL DEFLECTION @ CTR IS SUM OF DEFLECTIONS @ CTR.

USING APPENDIX "C"

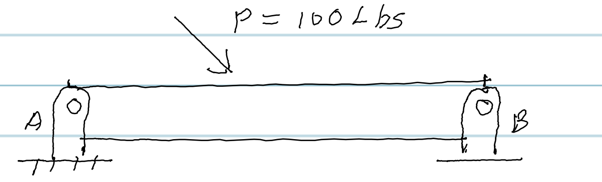
$$\Delta_{POINT\ LOAD} = -\frac{PL^3}{48EI}$$

TOTAL DEFLECTION $\Delta_{TOTAL} = \Delta_{POINT\ LOAD} + \Delta_u$

$$\Delta_T = \frac{-PL^3}{48EI} - \frac{5wL^4}{384EI}$$

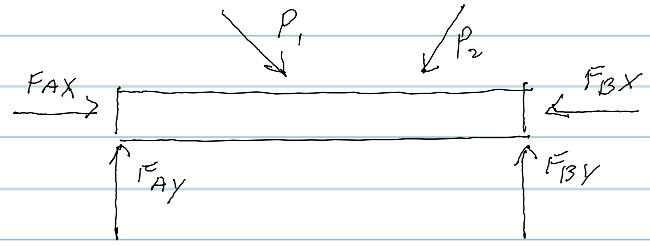
CH. 12.6 STATICALLY INDETERMINATE BEAMS & SHAFTS

STATICALLY INDETERMINATE



COUNT UNKNOWNS ON FBD
(M, F_x's, F_y's – EACH COMPONENT)

REDUNDANT FORCES



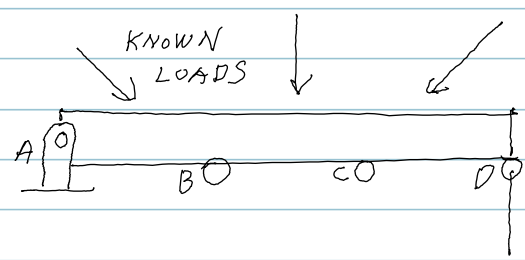
☐ CAN BE REMOVED & FBD REMAINS STABLE

# OF UNKNOWN'S	4
# OF EQUATIONS	3
# OF REDUNDANTS	1

☐ CHOICE OF REDUNDANTS:

F_{BX} , F_{AX} BUT ONLY REMOVE ONE!

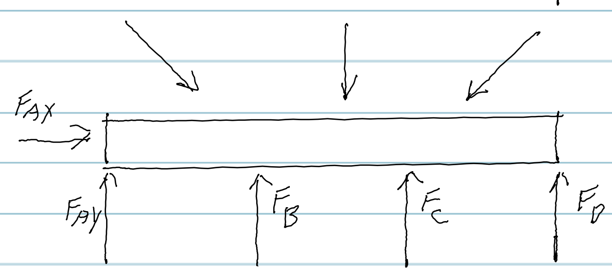
TEST YOUR UNDERSTANDING



______ # OF UNKNOWNS

______ # OF REDUNDANTS

______ CHOICE OF REDUNDANTS



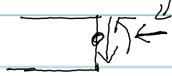
______ # OF UNKNOWNS YOU CAN REMOVE & FBD REMAINS STABLE

CH. 12.7 STATICALLY INDETERMINATE BEAMS + SHAFTS – INTEGRATION

NEED ADDITIONAL EQUATIONS

USE MATERIAL PROPERTIES

EXPANSION, ΔL – NORMAL STRESS $\sigma_N = E\epsilon$ (CH. 4.4 + SN pg 28)



TORQUE – $\tau = \frac{T_s L}{J G}$ MODULUS OF RIGIDITY (CH. 5.5 + SN pg. 41)

BENDING – $EI \frac{dv^2}{dx^2} = M(x)$

FOR BENDING – DEVELOP ELASTIC CURVE – DIRECT INTEGRATION, MACAULAY

CHOICES

DIRECT INTEGRATION – GOOD CONTINUOUS FUNCTIONS

MACAULAY FUNCTIONS – MUCH EASIER

PROCESS

DEVELOP ELASTIC CURVE

STATE BOUNDARY CONDITION

SUBSTITUTE EACH BOUNDARY CONDITION INTO ELASTIC CURVE

SOLVE SET OF SIMULTANEOUS BOUNDARY CONDITION EQUATIONS

GET C_1, C_2, + REDUNDANT UNKNOWNS

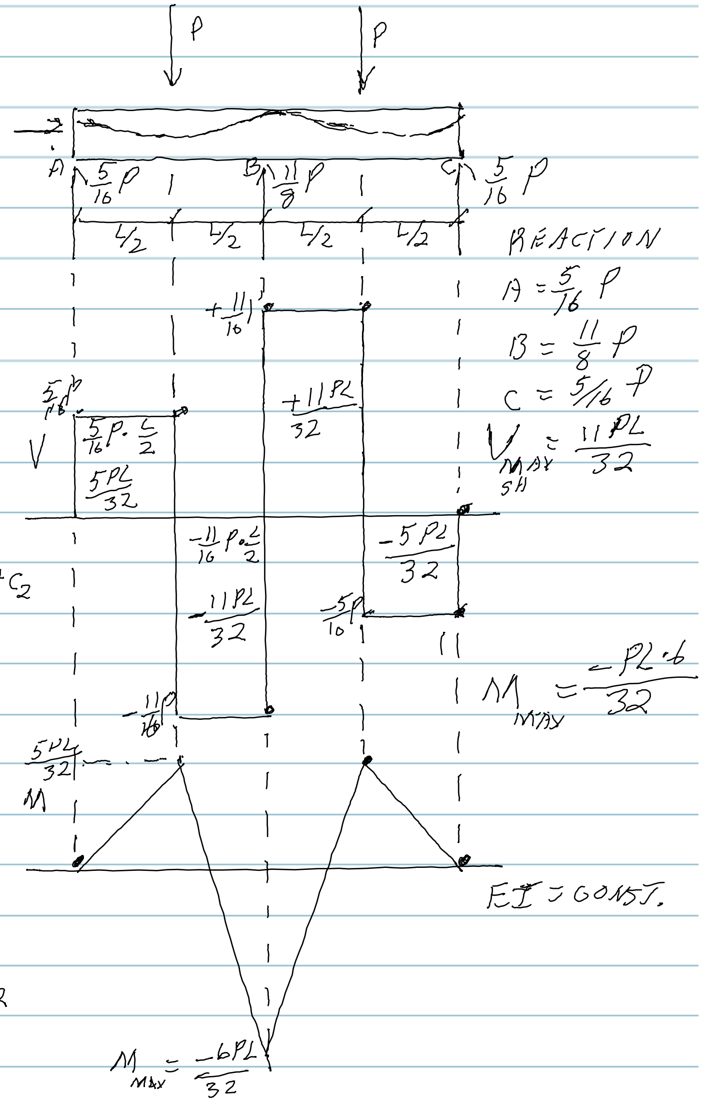
SOLVE FOR REMAINING UNKNOWNS W/ FBD

CH. 12.7 STATICALLY INDETERMINATE PROB. 12-101

GIVEN: TWO SPAN CONTINUOUS BEAM
EI = CONSTANT

REDUNDANT – 1 ST DEGREE

WRITE MOMENT EQUATION USING
MACAULAY FUNCTIONS $\quad M = P\langle x-a\rangle^1$



$$EI\frac{d^2v}{dx^2} = M(x) = +A\langle x-0\rangle^1 - P\langle x-\tfrac{L}{2}\rangle^1 + B\langle x-L\rangle^1 - P\langle x-\tfrac{3L}{2}\rangle^1$$

$$① \quad EI\frac{dv}{dx} = \frac{A}{2}\langle x-0\rangle^2 - \frac{P}{2}\langle x-\tfrac{L}{2}\rangle^2 + \frac{B}{2}\langle x-L\rangle^2 - \frac{P}{2}\langle x-\tfrac{3}{2}L\rangle^2 + C_1$$

$$② \quad EIv = \frac{A}{6}\langle x-0\rangle^3 - \frac{P}{6}\langle x-\tfrac{L}{2}\rangle^3 + \frac{B}{6}\langle x-L\rangle^3 - \frac{P}{6}\langle x-\tfrac{3L}{2}\rangle^3 + C_1x + C_2$$

BOUNDARY CONDITIONS:

a) $v(0) = 0 \quad @ \ x = 0$
b) $v(L) = 0 \quad @ \ x = L$
c) $\frac{dv}{dx}(L) = 0 \quad @ \ x = L$
d) $v(2L) = 0 \quad @ \ x = 2L$

$$2a \quad 0 = \frac{A}{6}\cancel{\langle 0-0\rangle^3}_{0} - \frac{P}{6}\cancel{\langle 0-\tfrac{L}{2}\rangle^3}_{0} + \frac{B}{6}\cancel{\langle 0-L\rangle^3}_{0} - \frac{P}{6}\cancel{\langle 0-\tfrac{3L}{2}\rangle^3}_{0} + C_1 \cdot 0 + C_2$$

$$C_2 = 0$$

$$2b \quad 0 = \frac{A}{6}\langle L-0\rangle^3 - \frac{P}{6}\langle L-\tfrac{L}{2}\rangle^3 + \frac{B}{6}\cancel{\langle L-L\rangle^3}_{0} - \frac{P}{6}\cancel{\langle L-\tfrac{3L}{2}\rangle^3}_{0} + C_1L$$

$$0 = \frac{AL^3}{6} - \frac{P}{6}\left(\frac{L}{2}\right)^3 + C_1L$$

$$1c \quad 0 = \frac{A}{2}\langle L\rangle^2 - \frac{P}{2}\langle L-\tfrac{L}{2}\rangle^2 + \frac{B}{2}\cancel{\langle L-L\rangle^2}_{0} - \frac{P}{2}\cancel{\langle L-\tfrac{3L}{2}\rangle^2}_{0} + C_1$$

$$0 = \frac{AL^2}{2} - \frac{P}{2}\left(\frac{L}{2}\right)^2 + C_1$$

$$2d \quad 0 = \frac{A}{6}\langle 2L-0\rangle^3 - \frac{P}{6}\langle 2L-\tfrac{L}{2}\rangle^3 + \frac{B}{6}\langle 2L-L\rangle^3 - \frac{P}{6}\langle 2L-\tfrac{3L}{2}\rangle^3 + C_1 \cdot 2L$$

$$0 = \frac{A}{6}(2L)^3 - \frac{P}{6}\left(\frac{3L}{2}\right)^3 + \frac{B}{6}(L^3) - \frac{P}{6}\left(\frac{L}{2}\right)^3 + 2LC_1$$

MATHCAD – PROB 12-101 ....mcd

$A = \frac{5}{16}P$

$B = \frac{11}{8}P$

$C_1 = -\frac{1}{32}PL^2$

$\Sigma F_y = 0$

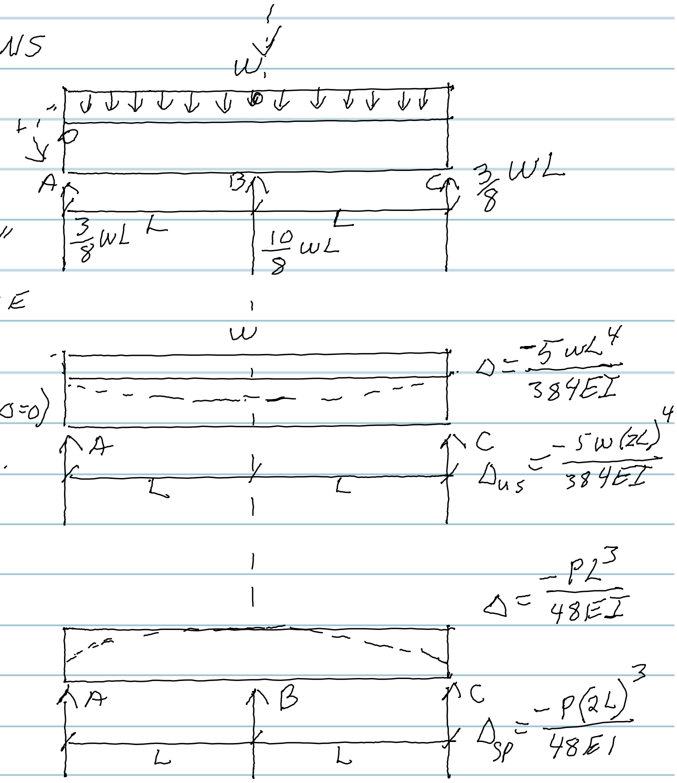
$-2P + \frac{5}{16}P + \frac{11}{8}P + C = 0$

$C = \frac{5}{16}P$

CH 12.9 STATICALLY INDETERMINATE BEAMS & SHAFTS

METHOD OF SUPERPOSITION - FIND REACTIONS

1) REMOVE REDUNDANT FORCE(S)
2) FIND DEFLECTION @ REDUNDANT
3) BREAK INTO LOAD CASES - APPENDIX "C"
4) REMOVE LOAD + REPLACE REDUNDANT FORCE
5) FIND DEFLECTION FOR ~~EACH~~ THIS CASE
6) WRITE DEFLECTION EQUATION (USUALLY $\Sigma\Delta = 0$)
7) SOLVE DEFLECTION EQ. FOR REDUNDANT FORCE.
8) PUT KNOWN REDUNDENT FORCE ON ORIGINAL FBD & SOLVE FOR REACTIONS.



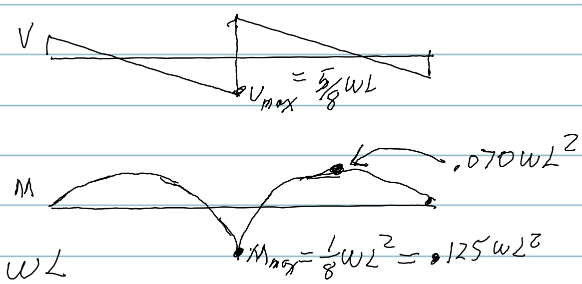
6) $\Delta_{us} = -\frac{5w(2L)^4}{384EI}$, $\Delta_{sp} = \frac{P(2L)^3}{48EI}$

7) $\Sigma\Delta = 0$

$$-\frac{5w(2L)^4}{384EI} + \frac{P(2L)^3}{48EI} = 0 \qquad B = P_{CASE}$$

$$\frac{B(2L)^3}{48\cancel{EI}} = \frac{5w(2L)^4}{384\cancel{EI}}$$

$$B = \frac{5w(2L)^4 \cdot 48}{(2L)^3 \cdot 384} = \frac{5w(2L) \cdot 48}{384} = \frac{10}{8}wL$$



8) $\Sigma M_A = 0$

$$+\left(\frac{10}{8}wL\right)L - (w2L)\cdot 2L + C \cdot 2L = 0$$

$$+\frac{10}{8}wL^2 - 2wL^2 + 2LC = 0$$

$$-\frac{6}{8}wL^2 + 2LC = 0$$

$$C = \frac{3}{8}wL$$

$\Sigma F_y = 0$

$$-w(2L) + \frac{10}{8}wL + \frac{3}{8}wL + A = 0$$

$$A = \frac{3}{8}wL$$